

Thomas Rotation and Polarised Light: A non-Abelian Geometric Phase in Optics

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Abstract

We describe a *non-Abelian* Berry phase in polarisation optics, suggested by an analogy due to Nityananda between boosts in special relativity and the effect of elliptic dichroism on polarised light. The analogy permits a simple optical realization of the non-Abelian gauge field describing Thomas rotation. We also show how Thomas rotation can be understood geometrically on the Poincaré sphere in terms of the Pancharatnam phase.

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1 Introduction

Shortly after Berry’s discovery of the geometric phase [1], the idea was generalized in various directions [2, 3, 4, 5]. Here we follow on the work of Wilczek and Zee [2], who noted that in quantum systems with N -fold degeneracy, the “phase” is no longer Abelian, but an element of $U(N)$, the unitary group in N dimensions. These non-Abelian phases have been studied in Nuclear Quadrupole resonance (NQR) [5].

There have been several demonstrations of the geometric phase [6, 7, 8, 9, 10, 11] in polarisation optics. However, *all* of these experiments deal with the *Abelian* phase that Berry’s work drew wide attention to. The purpose of this paper is to point out that the *non-Abelian* phase of Wilczek and Zee [2] can also be realised in polarisation optics. Since optical systems are much easier to set up than say, NQR experiments, we feel that our observation will aid experimental studies of the non-Abelian phase.

There has been some interest in the gauge theoretic aspects of Thomas rotation [12, 13], which can be abstractly [13] viewed as an $SO(3)$ gauge field living on the space of inertial observers. More concretely, Mathur [12] pointed out that the Thomas precession gauge field (TPGF) appears as a non-Abelian Berry phase due to Kramer’s degeneracy of the Dirac electron. Our purpose in this article is to present another concrete realization of the TPGF in polarisation optics. We believe our work may be of general interest, as it permits a simple optical demonstration of Thomas Rotation (TR). While our study is both theoretical and experimental, this paper is devoted to theoretical aspects. Experimental details will be reported elsewhere.

In section 2 we review the gauge theoretic aspects of Thomas Rotation as a prelude to realising it in optics in section 3. In section 4 we describe how

the phenomenon of TR can be understood in *purely optical terms* using the Pancharatnam Phase. Section 5 is a concluding discussion.

2 Thomas Rotation

As is well known in the special theory of relativity, *pure* boosts (changes of frame without rotation of the spatial axes) along different spatial directions do not commute with each other. In fact, the commutator of two infinitesimal boosts is a rotation and this leads to Thomas precession [14, 15]. We deal here with the *finite* version of this effect, the Thomas Rotation [16, 13]. Let u_1^μ , ($\mu = 0, 1, 2, 3$) be the four-velocity of an observer and $\Lambda_{21}, \Lambda_{32}$ and Λ_{13} a sequence of pure boosts (Λ is given explicitly in [13]) applied to u_1 . The sequence of pure boosts takes u_1 to u_2 ($= \Lambda_{21}u_1$), u_3 ($= \Lambda_{32}u_2$) and finally back to u_1 ($= \Lambda_{13}u_3$). Since the final four-velocity $\Lambda_{13}\Lambda_{32}\Lambda_{21}u_1$ is equal to the initial four-velocity u_1 , we say that the sequence of boosts *closes* (returns the observer to her original frame). In general, the product $\Lambda = \Lambda_{13}\Lambda_{32}\Lambda_{21}$ of boosts is not the identity, but $\Lambda = R$, where R is a rotation matrix representing the Thomas rotation. The rotation matrix R can be expressed as a path ordered product or Wilson loop:

$$R = P \exp \left[\oint A(u) du \right], \quad (1)$$

where the integral is over the geodesic triangle $u_1 - u_2 - u_3 - u_1$ on the unit hyperboloid \mathcal{H}^+ of four velocities $\mathcal{H}^+ = \{u^\mu | u \cdot u = 1, u^0 > 0\}$. The gauge field A describing Thomas rotation is a 3×3 antisymmetric matrix, whose components (in a convenient [13] global gauge choice over \mathcal{H}^+) are $A^i_j = (1 + u^0)^{-1}(du^i u_j - du_j u^i)$, where $i, j = 1, 2, 3$. A closed sequence of three boosts must lie in a plane [13]. This is no longer true for a closed

sequence of M (four or more) boosts, which involves M inertial observers with M four-velocities u_1, u_2, \dots, u_M . However, one can consider these M observers in triplets by triangulating the broken geodesic curve $C = u_1 - u_2 - \dots - u_M - u_1$ and applying the previous argument. The Thomas rotation is then expressed as a Wilson loop over C . In general C is a (closed, piecewise geodesic) *space* curve in \mathcal{H}^+ and path ordering in (1) becomes necessary, revealing the essentially *non-Abelian* nature of the gauge field. One can also consider the limit $M \rightarrow \infty$ of an infinite number of infinitesimal boosts. The integral in (1) is then over a continuous closed curve in \mathcal{H}^+ , which is in general nonplanar. Each boost distorts (alters the distances between points on) the celestial sphere of an observer due to the well known phenomenon of aberration of light [15, 17] from the sky. When the sequence of boosts closes, the distortions cancel and the net result is an undistorted but rotated celestial sphere, the rotation matrix being given by (1).

3 Analogue of Thomas Rotation in Optics

In order to realise the TPGF A_j^i in optics, we draw on an observation by Nityananda [18] regarding the effect of an anisotropic absorber on the polarisation of light. Let \hat{n} be a unit vector representing a general point on the Poincaré sphere. Suppose that light is passed through an absorber $A_{\hat{n}}$ that preferentially absorbs the polarisation state \hat{s} orthogonal to \hat{n} . If the incident light is in the state \hat{n} or \hat{s} , its polarisation state will not be affected by the material (although its intensity may diminish). Any other polarisation state \hat{p} on the Poincaré sphere can be resolved in terms of \hat{n} and \hat{s} and, since the \hat{s} component of \hat{p} is preferentially absorbed, will move towards \hat{n} along the great circle [19] joining \hat{p} to \hat{n} . The key observation due to Nityananda [18]

is that the effect of an absorber $A_{\hat{n}}$ on the Poincaré sphere is identical to the effect of a pure Lorentz boost in the direction \hat{n} on the celestial sphere of an observer in special relativity.

The optical element $A_{\hat{n}}$ discussed above is purely absorbing (dichroic) in the sense that it introduces only a relative attenuation between two orthogonal states and not a relative phase. Dichroic elements $A_{\hat{n}}$ distort the Poincaré sphere. There are also birefringent elements $R_{\hat{n}}$ (retarders, such as wave plates), which introduce a *pure* phase difference between two orthogonal states, and rigidly rotate the Poincaré sphere about an axis \hat{n} . The effect of absorbers and retarders on the Poincaré sphere is qualitatively different. Yet, by exploiting the analogy with special relativity, we see that a sequence of absorptions along different directions can give rise to a net rigid rotation of the Poincaré sphere. A sequence of elliptic dichroids can result in a net elliptic birefringence. *This is the optical analogue of Thomas rotation.*

Before developing this analogy in quantitative terms, we briefly clarify our terminology, which some readers may find unfamiliar. Traditionally, optical elements which cause a rotation of the Poincaré sphere about the x or y axis are called birefringent, whereas those which generate a rotation about the z axis are called optically active. There is no standard terminology for an element which causes a rotation of the Poincaré sphere about an arbitrary axis. Since these are merely rotations about different axes, there is no fundamental difference between them. We adopt a generalised terminology and refer to all of these as elliptically birefringent elements. Special cases of elliptic birefringence are linear (ordinary birefringence), and circular (optical activity). We also use the word dichroism in the same general sense to refer to elliptic dichroism and not just linear or circular dichroism.

Consider an absorbing optical device $A_{\hat{n}}$ whose effect on a state $|n\rangle$

is to reduce its amplitude to $e^{-\alpha_1}|n\rangle$ and whose effect on the state $|s\rangle$ orthogonal to $|n\rangle$ is to reduce its amplitude to $e^{-\alpha_2}|s\rangle$ (we assume $\alpha_2 > \alpha_1$, so state $|s\rangle$ is preferentially absorbed). It is convenient to introduce the relative and overall absorption coefficients $\alpha = \alpha_2 - \alpha_1$ and $\alpha_0 = (\alpha_1 + \alpha_2)/2$ respectively. Since we are not interested in the overall intensity and phase of the light beam (these are not represented on the Poincaré sphere), we do not need to normalise our state vectors. A general (unnormalised) state $|p\rangle$ can be expanded in the orthonormal basis $\{|n\rangle, |s\rangle\}$: $|p\rangle = |n\rangle + z|s\rangle$, where $z = \tan(\theta/2)e^{i\phi}$ with θ the angle between \hat{p} and \hat{n} on the Poincaré sphere (the “colatitude”) and ϕ the “longitude”. The effect of $A_{\hat{n}}$ on $|p\rangle$ is to transform it to $|p'\rangle = |n\rangle + e^{-\alpha}z|s\rangle$ (where we have discarded an overall factor which does not affect the polarisation state). On the Poincaré sphere \hat{p}' has the same “longitude” as \hat{p} and makes an angle θ' with \hat{n} where $\tan(\theta'/2) = e^{-\alpha} \tan(\theta/2)$. A little algebra shows that $\cos\theta' = (\cos\theta + \tanh\alpha)/(1 + \cos\theta \tanh\alpha)$. Letting $\beta = \tanh\alpha$, we notice that this is identical to the aberration formula [15]. The relative absorption α plays the role of rapidity in the analogue relativistic system and β is the velocity in units of the speed of light [20]. An absorber $A_{\hat{n}}(\alpha)$ with relative absorption α corresponds to a boost $\Lambda_{\hat{n}}(\alpha)$ in the \hat{n} direction with rapidity α . By applying successive boosts $\Lambda_{\hat{n}}(\alpha)$ to the initial four-velocity u_1^μ (say $(1, 0, 0, 0)$), one can trace the path of the equivalent relativistic system [21] on \mathcal{H}^+ .

4 Optical View of Thomas Rotation

We now show how Thomas rotation can be understood in purely optical terms, using the Pancharatnam phase [22, 23, 24]. Let a light beam pass through a sequence of three absorbers $A_{\hat{n}_1}, A_{\hat{n}_2}$ and $A_{\hat{n}_3}$ which closes (the

combination $A = A_{\hat{n}_3}A_{\hat{n}_2}A_{\hat{n}_1}$ preserves distances between points on the Poincaré sphere). Consider the set of points \mathcal{C} on the great circle through \hat{n}_1 and \hat{n}_2 . Since these points are “dragged” *along* \mathcal{C} by both $A_{\hat{n}_1}$ and $A_{\hat{n}_2}$, the set \mathcal{C} is left invariant by both these absorbers.

Lemma 1: If the sequence $A_{\hat{n}_1}, A_{\hat{n}_2}, A_{\hat{n}_3}$ closes, \hat{n}_3 must lie on the great circle \mathcal{C} .

Proof: If \hat{n}_3 is not on \mathcal{C} , it must lie on one of the hemispheres into which \mathcal{C} divides the Poincaré sphere. $A_{\hat{n}_1}$ and $A_{\hat{n}_2}$ leave \mathcal{C} invariant, but the effect of $A_{\hat{n}_3}$ is to drag all points of \mathcal{C} into the hemisphere containing \hat{n}_3 . Antipodal points of \mathcal{C} (which are separated by a distance π) will no longer be antipodal since they are in the same hemisphere. This contradicts the assumption that the sequence of three absorbers closes. It follows that the points $\hat{n}_1, \hat{n}_2, \hat{n}_3$ lie on a great circle.

\mathcal{C} is left invariant by all the three absorptions and therefore by their composition $A = A_{\hat{n}_3}A_{\hat{n}_2}A_{\hat{n}_1}$. Let \hat{n} and \hat{s} be the two points on the Poincaré sphere which satisfy $\hat{n} \cdot \hat{n}_i = \hat{s} \cdot \hat{n}_i = 0$ for $i = 1, 2, 3$.

Lemma 2: \hat{n} and \hat{s} are returned to their original positions by the sequence $A = A_{\hat{n}_3}A_{\hat{n}_2}A_{\hat{n}_1}$.

Proof: Since \mathcal{C} is invariant under the action of $A_{\hat{n}_3}, A_{\hat{n}_2}$ and $A_{\hat{n}_1}$, points on the Poincaré sphere do not cross \mathcal{C} . Since \hat{n} is the only point on its hemisphere which is $\pi/2$ away from \mathcal{C} , \hat{n} must be mapped to itself by A . Similarly, \hat{s} , which is antipodal to \hat{n} , is also mapped to itself.

Lemma 3: The action of A on the Poincaré sphere is a rotation about the $\hat{n} - \hat{s}$ axis.

Proof: Under the action of $A_{\hat{n}_3}A_{\hat{n}_2}A_{\hat{n}_1}$, \hat{n} and \hat{s} are dragged along great circles towards \hat{n}_1, \hat{n}_2 and \hat{n}_3 as shown in Fig.1. (The closure of this sequence of geodesic arcs is guaranteed by *Lemma 2*). From Pancharatnam’s theorem

[22, 23], we see that the state \hat{n} picks up a geometric phase $\Omega/2$, where Ω is the solid angle subtended by the geodesic triangle traced by the state \hat{n} as it traverses the geodesic arcs. From Fig.1, note that states \hat{n} and \hat{s} pick up equal and opposite geometric phases $\pm\Omega/2$. It follows from the invariance of C that A introduces no relative attenuation between states \hat{n} and \hat{s} . This is the signature of elliptic birefringence in the \hat{n} - \hat{s} direction. The effect of A on the Poincaré sphere is a rotation through an angle Ω about the $\hat{n} - \hat{s}$ axis. It is elementary to verify that Ω is equal to the Thomas rotation angle [16].

It now follows that the rotation of the Poincaré sphere due to a closed sequence of M absorbers is given by the Wilson loop (1) over the curve C . A sequence of M absorbers can be represented as a piecewise geodesic closed space curve C in \mathcal{H}^+ . By triangulation, one can reduce the traversal of C into a sequence of traversals of triangles. Each triangle causes a rotation about some axis as discussed above in *Lemma 3*. Since C is not in general planar, the rotations caused by different triangles are about *different* axes. As a result, one has to *compose* the sequence of rotations (and not just add the rotation angles, as one does in the Abelian case). The final answer is a path ordered product of individual rotations (1).

5 Conclusion

A particular case of the optical effect discussed here has been noticed earlier: linear dichroism can lead [25, 26] to optical activity (circular birefringence). Kitano et al [26] study Lorentz group Berry phases, but since they use only *linear absorbers*, they only explore an *Abelian* phase. As Zee [27] pointed out in the context of the NQR experiment of Tycko [28], this amounts to exploring a one parameter *Abelian subgroup* of the full *non-Abelian gauge*

group. There is also a difference between our approach and Ref.[26], which views the linear absorbers as “squeezing” the polarisation ellipse similar to Berry’s discussion of the time dependent oscillator [29]. The relevant group is then $SO(2, 1)$ and the Abelian phase, a $U(1)$ phase. Our approach is based on *distortions of the Poincaré sphere*. The relevant group here is $SO(3, 1)$, the Lorentz group in $3 + 1$ dimensions, which results in a *Non-Abelian* gauge group $SO(3)$.

We have dealt with purely absorbing elements giving rise to a net birefringence. In general optical elements will have both absorption and birefringence. In the language used in the Berry phase literature, the birefringence is the dynamic “phase” and has to be “subtracted out” before one sees the geometric “phase”. In our exposition we chose optical elements which are purely absorbing so that the dynamic “phase” is zero. This is similar in spirit to recent demonstration of the Pancharatnam phase from pure projections [10, 11] where the dynamic phase is manifestly zero. Indeed, in the limit of infinite relative absorption ($\alpha \rightarrow \infty$) absorbers become projectors. It should be borne in mind that this limit is highly singular, corresponding in the relativistic language, to boosting a frame to the speed of light.

There is a mathematical similarity between the effect described here and the Pancharatnam effect. Both demonstrate the mathematical phenomenon of anholonomy of which there are numerous instances in physics [5]. In both cases, the mathematical structure explored is a connection on a fiber bundle $(\mathcal{P}, \mathcal{B}, \Pi)$, where \mathcal{P} is the total space, \mathcal{B} , the base space and Π a projection from \mathcal{P} to \mathcal{B} . $\Pi^{-1}(b)$ is called the fiber over b . A connection on a fiber bundle gives a rule for comparing points on neighboring fibers. In general, the connection is not integrable. When one parallel transports a point $p \in \Pi^{-1}(b)$ along a closed curve in \mathcal{B} , one returns to the same fiber,

but not in general to the same point on that fiber. This is a reflection of the curvature of the connection or the nonintegrability of the rule for comparison of points. In the context of the Pancharatnam phase, the base space \mathcal{B} is the Poincaré sphere S^2 , the fiber is the phase $U(1)$ and the total space \mathcal{P} is S^3 . In the case of present interest, the base space is the unit hyperboloid \mathcal{H}^+ of four-velocities and the fiber is the rotation group $SO(3)$ and the total space is the Lorentz Group.

The optical realization of TPGF is a simple analogue experiment which can be easily performed. In the Dirac electron realization of TPGF [12], relativistic effects like TR are small under normal laboratory conditions. In the analogue optical system, attaining “relativistic” speeds is quite easy. One just uses an absorber with a high relative absorption. One is limited only by one’s ability to detect the residual intensity of preferentially absorbed polarisation. In preliminary studies, we have measured Thomas rotation angles of upto 50° in the optics laboratory. A possible application of the ideas described in this paper is the design of achromatic (wavelength independent) birefringent elements. Because of the geometric character of the effect, birefringent devices (optical rotators or wave plates) constructed from dichroic elements as described above will introduce the same phase difference independent of wavelength, provided of course, the dichroism is not wavelength dependent in the frequency bandwidth of interest.

Historically, polarised light has played a significant role in elucidating the field of Geometric phases. We believe our paper goes further in this direction by showing how *Non-Abelian* phases can also be seen in optics.

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- [20] In passing, we note that the aberration formula is considerably simplified by using an angular variable σ defined by $\cos \theta = \tanh \sigma$. The aberration formula then reads $\sigma' = \sigma + \alpha$.
- [21] If one forms the (impure) density matrix $\rho = u^0 + \vec{u} \cdot \vec{\sigma}$, the components (u^0, \vec{u}) have an optical interpretation in terms of the Stokes [19] parameters (I, Q, U, V) of a partially polarised light beam described by that density matrix.

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Figure Caption

Figure 1 : The Poincaré sphere representation of the trajectories of the two orthogonal elliptic polarisations \hat{n} and \hat{s} . These trajectories are mirror images of each other in the plane containing \hat{n}_1 , \hat{n}_2 and \hat{n}_3 . Note that by Pancharatnam's theorem, these two polarisations pick up equal and opposite geometric phases $\pm\Omega/2$. This corresponds to a rotation of the Poincaré sphere by an angle Ω about the axis passing through the orthogonal elliptic states \hat{n} and \hat{s} .

